ON THE THEORY OF DEFORMATION OF MICRO-INHOMOGENEOUS BODIES AND ITS RELATION WITH THE COUPLE STRESS THEORY OF ELASTICITY

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The task of a complete statistical description of the state of a micro-inhomogeneous body which is subjected to the action of arbitrary surface forces is reduced to two problems: (1) the problem of the generalized couple stress theory of elasticity and (2) the problem of the micro-inhomogeneous theory of elasticity in conditions of a macroscopically homogeneous state of deformation. It is also shown that the solution of the boundary problem of the generalized couple stress theory of elasticity determines mean displacements of the examined micro-inhomogeneous body. Some variants of formulating of boundary problems of the couple stress theory of elasticity were also considered and their relation with the theory of deformation of micro-inhomogeneous bodies was determined.

1. Let us consider the deformation of a micro-inhomogeneous elastic body in which stresses τ_{ij} and deformations e_{ij} (displacements w_i) are related by Hooke's law

$$\tau_{ij} = c_{ijlm} e_{lm} = c_{ijlm} \frac{\partial w_l}{\partial x_m} \tag{1.1}$$

with a tensor of moduli of elasticity c_{ijlm} representing a statistically homogeneous and isotropic random tensor field [1]. Determination of statistical characteristics for the field of displacements is reduced to the solution of stochastic differential equations

$$\frac{\partial}{\partial x_j}\left(c_{ijlm} \frac{\partial w_l}{\partial x_m}\right) = 0 \tag{1.2}$$

with corresponding deterministic boundary conditions.

The solution of problem (1.2) in the case of a macroscopically homogeneous state of deformation ε_{jk} = const [2] for fluctuations w_i' of the vector of displacements gives

$$w_i' = \varphi_{ijk}(x_s)\, \varepsilon_{jk} \tag{1.3}$$

Here

$$w_i' = w_i - u_i, \qquad u_i = \langle w_i \rangle, \qquad \varepsilon_{jk} = \langle e_{jk} \rangle = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) \tag{1.4}$$

Here and in the following the statistical mean of the corresponding function is indicated by angular brackets and deviations from the average by primes. The tensor φ_{ijk} is determined by Green's tensor of the problem of the homogeneous theory of elasticity and by the tensor c'_{ijlm} of deviations of moduli of elasticity.

Considering now a macroscopically inhomogeneous state of deformation of a body and assuming that characteristic dimensions of inhomogeneities are small in comparison with the distance at which changes in macroscopic deformations ε_{jk} are noticeable, we shall accept that the relation (1.3) defines fluctuations of displacements w_i' of a macroscopic volume element also in the case of an inhomogeneous field $\varepsilon_{jk}(x_s)$. In other words, the relationship (1.3), exact for ε_{jk} = const, will also be applied for weakly varying (in comparison with fluctuations w_i') fields $\varepsilon_{jk}(x_s)$.

Computing according to this assumption the mean value W of the specific potential energy of deformation

$$W = {}^1/_2 \langle \tau_{ij} e_{ij} \rangle = {}^1/_2 \langle \tau_{ij} \rangle \langle e_{ij} \rangle + {}^1/_2 \langle \tau_{ij}' e_{ij}' \rangle \tag{1.5}$$

we find

$$W = {}^1/_2\, a_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + {}^1/_2\, b_{ijklmn} \varkappa_{ijk} \varkappa_{lmn} + d_{ijklmn} \varepsilon_{ij} \varkappa_{klm} \tag{1.6}$$

where $\varkappa_{ijk}$ is the gradient of the strain tensor

$$\varkappa_{ijk} = \frac{\partial \varepsilon_{jk}}{\partial x_i} \tag{1.7}$$

In the considered case of a statistically isotropic field c_{ijlm}, the tensor d_{ijklm} vanishes and the tensors a_{ijkl} and b_{ijklmn} are isotropic tensors. Taking into account the existing symmetry of these tensors, the relation (1.6) can be presented as

$$W = {}^1/_2 \lambda \varepsilon_{ii} \varepsilon_{kk} + \mu \varepsilon_{kl} \varepsilon_{kl} + \eta_1 \varkappa_{iik} \varkappa_{kjj} + \eta_2 \varkappa_{ijj} \varkappa_{ikk} + \\ + \eta_3 \varkappa_{iik} \varkappa_{jjk} + \eta_4 \varkappa_{ijk} \varkappa_{ijk} + \eta_5 \varkappa_{ijk} \varkappa_{kji} \tag{1.8}$$

The first two terms of the relation (1.8) include the work of mean stresses on mean strains (first term of (1.5)) and a part of the work of fluctuations of stresses on fluctuations of strains (a part of the second term of (1.5)) which corresponds to the energy of micro-deformations [3], occurring at a macroscopically homogeneous state of deformation. The remaining terms of the relation (1.8) determine the contribution brought to the mean energy of deformation by the directional effect of gradients of macrodeformations on the distribution of stress and strain fluctuations.

The basic hypothesis on validity of the relationship (1.3) in an inhomogeneous field $\varepsilon_{jk}(x_s)$ reduces the problem of deformation of mean displacements $\langle w_i \rangle$ for the considered micro-inhomogeneous body (for definiteness we shall assume that w_i is the solution of the boundary value problem of the equations (1.2) for specified deterministic forces q_i acting on the surface of the body) to the problem of determination of displacements u_i of a homogeneous elastic body of the same shape and subjected to forces $q_i(x_s)$ given on the surface s of the body and whose density of the potential energy of deformation is determined by the relation (1.8). The displacements u_i determined in this way are only approximately equal to $\langle w_i \rangle$ and the approximation is the better, the more accurately the relationship (1.3) is fulfilled.

2. The boundary value problem for determination of the vector u_i is obtained from the Lagrange's variational principle

$$\int_{(v)} \delta W\,(\varepsilon_{ij},\ \varkappa_{ijk})\,dv = \int_{(s)} q_i \delta u_i\,ds \tag{2.1}$$

Introducing tensors σ_{ij} and μ_{ijk} as generalized forces for corresponding generalized displacements ε_{ij} and $\varkappa_{ijk}$

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \qquad \mu_{ijk} = \frac{\partial W}{\partial \varkappa_{ijk}} \tag{2.2}$$

we have

$$\delta W = \sigma_{ij}\delta\varepsilon_{ij} + \mu_{ijk}\delta\varkappa_{ijk}$$

Expressing ε_{ij} and $\varkappa_{ijk}$ by means of displacements u_i, we can present δW in the form

$$\delta W = \frac{\partial}{\partial x_k}\left[\left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i}\right)\delta u_j\right] - \frac{\partial}{\partial x_k}\left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i}\right)\delta u_j + \frac{\partial}{\partial x_i}\left(\mu_{ijk}\frac{\partial \delta u_j}{\partial x_k}\right)$$

Hence

$$\int_{(v)} \delta W\,dv = \int_{(s)} \left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i}\right) n_k \delta u_j\,ds - \\ - \int_{(v)} \frac{\partial}{\partial x_k}\left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i}\right)\delta u_j\,dv + \int_{(s)} \mu_{ijk} n_i \frac{\partial \delta u_j}{\partial x_k}\,ds \tag{2.3}$$

Variation $\partial \delta u_j / \partial x_k$ can be presented as

$$\frac{\partial \delta u_j}{\partial x_k} = n_k \delta v_j + \varepsilon_{skl}\varepsilon_{mrl} n_s n_m \frac{\partial \delta u_j}{\partial x_r} \qquad \left(\delta v_j = \frac{\partial \delta u_j}{\partial x_s} n_s\right) \tag{2.4}$$

Here δv_j is the part of the variation independent of δu_j and ε_{ijk} is the antisymmetrical unit pseudo-tensor; its components are equal to +1 (−1) if i, j and k form an even (odd) permutation of numbers 1, 2 and 3, and are equal to zero if any two indices are the same. Using (2.4) we find

$$\mu_{ijk} n_i \frac{\partial \delta u_j}{\partial x_k} = \mu_{ijk} n_i n_k \delta v_j + \varepsilon_{mrl} n_m \frac{\partial}{\partial x_r}\left(\varepsilon_{skl}\mu_{ijk} n_i n_s \delta u_j\right) - \\ - \frac{\partial}{\partial x_r}\left(\mu_{ijr} n_i n_r - \mu_{ijk} n_i n_r\right) n_k \delta u_j \tag{2.5}$$

Substituting (2.3) and (2.5) into (2.1) we obtain

$$\int_{(s)} \left\{\left[\sigma_{jk} - \frac{\partial}{\partial x_i}\left(\mu_{ijk} + \mu_{lij} n_l n_k - \mu_{ljk} n_l n_i\right)\right] n_k - q_j\right\}\delta u_j\,ds - \\ - \int_{(v)} \frac{\partial}{\partial x_k}\left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i}\right)\delta u_j\,dv + \int_{(s)} \mu_{ijk} n_i n_k \delta v_j\,ds + \tag{2.6}$$

$$+ \int_{(s)} \varepsilon_{mrl} n_m \frac{\partial}{\partial x_r} (\varepsilon_{skl} \mu_{ijk} n_i n_s \delta u_j) \, ds = 0$$

For a smooth surface s, according to Stokes theorem, the last integral in (2.6) is equal to zero and, therefore, utilizing the independence of variations, we find from (2.6) equations of equilibrium

$$\frac{\partial}{\partial x_k} \left(\sigma_{jk} - \frac{\partial \mu_{ijk}}{\partial x_i} \right) = 0 \tag{2.7}$$

and boundary conditions for s

$$\left[\sigma_{jk} - \frac{\partial}{\partial x_i} (\mu_{ijk} + \mu_{lij} n_l n_k - \mu_{ljk} n_l n_i) \right] n_k = q_j$$
$$\mu_{ijk} n_i n_k = 0 \tag{2.8}$$

If the surface s is piece-wise smooth, then from the last integral in (2.6) we also obtain conditions at the kink

$$[[\mu_{ijk} n_i \nu_k]] = 0 \tag{2.9}$$

where $\nu_k = \varepsilon_{kls} t_l n_s$, and t_l are components of the unit vector which is tangential to the kink. The symbol $[[\ldots]]$ indicates the quantity shown in the brackets presents the difference between the values corresponding to different sides of the kink.

It should be taken that (in conformity with the Lagrange principle) the quantities σ_{ij} and μ_{ijk} are expressed by displacements according to (2.2).

3. The boundary value problem (2.7) to (2.9) for determination of u_i represents a generalization of the couple stress theory of elasticity for a special form of boundary conditions (at the boundary, only surface forces q_j are different from zero). The couple stress theory of elasticity is usually understood [4 to 6] as the theory according to which the deformation energy W is determined by the strain tensor and the gradient of the rotation (ω_i) vector determining the rotation of an elementary volume of the body. It can be shown that this theory is equivalent to the theory in which W represents a function of the strain tensor and of the antisymmetrical part $\varkappa_{[ij]k}$ of the tensor $\varkappa_{ijk}$, i.e. the theory in which W is expressed in the form

$$W = W(\varepsilon_{ij}, \varkappa_{[ij]k}) \tag{3.1}$$

The boundary value problem (2.7) to (2.9) corresponds to the energy

$$W = W(\varepsilon_{ij}, \varkappa_{ijk}) \tag{3.2}$$

where all components of the gradient of the tensor of deformations are taken into account. Some problems concerning the theory with energy (3.2) were examined in papers [7 and 8].

In this manner the problem of determination of mean displacements $u_j(x_s)$ of a micro-inhomogeneous medium (1.1) in a state of macroscopically inhomogeneous deformation is brought to the problem of the generalized couple stress (homogeneous) theory of elasticity (2.7) to (2.9). Let us assume that this problem is solved and the functions $u_j(x_s)$ and consequently also $\varepsilon_{jk}(x_s)$ are found. Then, the fluctuations w_i' of displacements of the considered micro-inhomogeneous medium will be found from the relations (1.3) and they will be fully determined, if the functions $\varphi_{ijk}(x_s)$ from (1.3) are known, i.e. if the problem of the fluctuations distribution at a macroscopically homogeneous deformation state of the body is solved. From the mean values $u_j(x_s)$ and fluctuations $w_i'(x_s)$ of displacements, moments of any order of the vector of displacements and of tensors of deformations and strains can be constructed [2] i.e. a complete statistical description of the deformed medium [1] can be given. Consequently, for a complete statistical description of the state of a micro-inhomogeneous body (1.1) which is subjected to the action of arbitrary surface forces $q_j(x_s)$, a successive solution of two problems is required: (1) of the problem of a generalized couple stress theory of elasticity of a homogeneous body whose solution determines mean displacements, and (2) of the problem of the inhomogeneous theory of elasticity with special boundary conditions, which guarantee a state of macroscopically homogeneous deformation; the solution of this problem (together with the solution of the previous one) determines fluctuations of displacements.

4. We shall now show that if we do not take into account the contribution to the energy (1.8) brought by the symmetrical part of the gradient of the deformation tensor, then the problem of determination of the displacement u_i coincides with the boundary value problem of the couple stress theory of elasticity presented in the usual form.

If in (1.8) only the antisymmetrical part $\varkappa_{[ij]k}$ of the tensor $\varkappa_{ijk}$, is taken into account, then W is expressed as follows

$$W = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ik}\varepsilon_{ik} + 2\mu l^2(\varkappa_{ij}\varkappa_{ij} + \eta\varkappa_{ij}\varkappa_{ji})$$
$$\left(l^2 = \frac{1}{8\mu}(\eta_2 + 2\eta_4 - \eta_5),\ \eta = -\frac{\eta_2}{8\mu l^2}\right) \qquad (4.1)$$

The tensors $\varkappa_{ij}$ and $\varkappa_{[ij]k}$ are interrelated by

$$\varkappa_{ij} = \varepsilon_{jlm}\varkappa_{lmi} = \varepsilon_{jlm}\varkappa_{[lm]\,i},\quad \varkappa_{[lm]\,i} = {}^1/_2\varepsilon_{jlm}\varkappa_{ij} \qquad (\varkappa_{ii} = 0) \qquad (4.2)$$

For the tensors

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \qquad \mu_{ij} = \frac{\partial W}{\partial \varkappa_{ij}}$$

which represent generallized forces displacements ε_{ij} and $\varkappa_{ij}$, generallized, from (4.1) we have

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \qquad \mu_{ij} = 4\mu l^2(\varkappa_{ij} + \eta\varkappa_{ji})$$

The energy variation

$$\delta W = \sigma_{ij}\delta\varepsilon_{ij} + \mu_{ij}\delta\varkappa_{ij}$$

can be expressed as

$$\delta W = \frac{\partial}{\partial x_l}\left[\left(\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\frac{\partial \mu_{ij}}{\partial x_i}\right)\delta u_k\right] -$$
$$-\frac{\partial}{\partial x_l}\left(\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\frac{\partial \mu_{ij}}{\partial x_i}\right)\delta u_k + \frac{\partial}{\partial x_i}(\mu_{ij}\delta\omega_j) \qquad (4.3)$$

where ω_j is the rotation vector

$$\omega_j = \frac{1}{2}\varepsilon_{jlm}\frac{\partial u_m}{\partial x_l} \tag{4.4}$$

Then from (4.3) we have

$$\begin{aligned}\int\limits_{(v)}\delta W\,dv = \int\limits_{(s)}\left(\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\frac{\partial\mu_{ij}}{\partial x_i}\right)n_l\delta u_k\,ds - \\ - \int\limits_{(v)}\frac{\partial}{\partial x_l}\left(\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\frac{\partial\mu_{ij}}{\partial x_i}\right)\delta u_k\,dv + \int\limits_{(s)}\mu_{ij}n_i\delta\omega_j\,ds\end{aligned} \tag{4.5}$$

Separating those variations $\delta\psi_j = \delta\,(\omega_j - \omega_s n_s n_j)$, which are independent of δu_k, we present the integrand of the last integral in (4.5) as

$$\begin{aligned}\mu_{ij}n_i\delta\omega_j = (\mu_{ij}n_i - \mu_{(n)}\,n_j)\,\delta\psi_j - \frac{1}{2}\varepsilon_{klj}\frac{\partial\mu_{(n)}}{\partial x_j}n_l\delta u_k + \\ + \varepsilon_{ljk}n_l\frac{\partial}{\partial x_j}\left(\frac{1}{2}\mu_{(n)}\,\delta u_k\right), \qquad \mu_{(n)} = \mu_{ij}n_in_j\end{aligned} \tag{4.6}$$

Substituting (4.5) and (4.6) into (2.1), considering the surface s to be piece-wise smooth and using Stokes formula, we find the equations of equilibrium

$$\frac{\partial}{\partial x_l}\left(\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\frac{\partial\mu_{ij}}{\partial x_i}\right) = 0 \tag{4.7}$$

the boundary conditions

$$\left[\sigma_{kl} + \frac{1}{2}\varepsilon_{klj}\left(\frac{\partial\mu_{ij}}{\partial x_i} - \frac{\partial\mu_{(n)}}{\partial x_j}\right)\right]n_l = q_k, \quad (\mu_{ij}n_i - \mu_{(n)}\,n_j) = 0 \tag{4.8}$$

and the conditions at the kinks

$$[[\mu_{(n)}]] = 0 \tag{4.9}$$

The relations (4.7) to (4.9) coincide (with boundary conditions adequately chosen) with the equations for the boundary value problem of the couple stress theory of elasticity presented in the usual form [5].

5. The tensor $\varkappa_{ijk}$ can be resolved into component tensors which have a definite geometric meaning and stand on their own. Different tensors, isolated from $\varkappa_{ijk}$, may play different roles in deformation processes of micro-inhomogeneous bodies, therefore such a tensor separation appears to be appropriate. In particular, any tensor $\varkappa_{ijk}$ symmetrical in the indices j and k can be presented in the form

$$\varkappa_{ijk} = \gamma_{ijk} + {}^1/_3\,\varepsilon_{ijl}\varkappa_{kl} + {}^1/_3\,\varepsilon_{ikl}\varkappa_{jl}, \qquad \gamma_{ijk} = {}^1/_3\,(\varkappa_{ijk} + \varkappa_{jki} + \varkappa_{kij})$$

Here $\varkappa_{ij}$ is determined by the relations (4.2). Then, introducing tensors

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}}, \qquad \mu_{ij} = \frac{\partial W}{\partial \varkappa_{ij}}, \qquad m_{ijk} = \frac{\partial W}{\partial \gamma_{ijk}}$$

we shall present the energy variation

$$\delta W = \sigma_{ij}\delta\varepsilon_{ij} + \mu_{ij}\delta\varkappa_{ij} + m_{ijk}\delta\gamma_{ijk}$$

as

$$\delta W = \frac{\partial}{\partial x_j}\left\{\left[\sigma_{ij} - \frac{\partial}{\partial x_k}\left(\frac{1}{2}\varepsilon_{jil}\mu_{kl} + m_{jik}\right)\right]\delta u_i\right\} -$$

$$- \frac{\partial}{\partial x_j}\left[\sigma_{ij} - \frac{\partial}{\partial x_k}\left(\frac{1}{2}\varepsilon_{jil}\mu_{kl} + m_{jik}\right)\right]\delta u_i + \frac{\partial}{\partial x_k}\left(\mu_{kl}\delta\omega_l + m_{jik}\frac{\partial \delta u_i}{\partial x_j}\right)$$

Here ω_l is the rotation vector (4.4). Hence

$$\int_{(v)} \delta W\, dv = -\int_{(v)} \frac{\partial}{\partial x_j}\left(\sigma_{ij} + \frac{1}{2}\varepsilon_{ijl}\frac{\partial \mu_{kl}}{\partial x_k} - \frac{\partial m_{ijk}}{\partial x_k}\right)\delta u_i\, dv +$$

$$+ \int_{(s)} \left(\sigma_{ij} + \frac{1}{2}\varepsilon_{ijl}\frac{\partial \mu_{kl}}{\partial x_k} - \frac{\partial m_{ijk}}{\partial x_k}\right) n_j \delta u_i\, ds + \int_{(s)} \mu_{kl} n_k \delta\omega_l ds + \int_{(s)} m_{ijk} n_k \frac{\partial \delta u_i}{\partial x_j} ds \tag{5.1}$$

The independent variations on the surface s are: variations of displacements δu_i, variations of the tangential component of the rotation vector $\delta\psi_l = \delta(\omega_l - \omega_s n_s n_l)$ and variations of the relative elongation $\delta(\varepsilon_{ls} n_l n_s)$ normal to the boundary. Separating independent variations, we shall write the integrand of the penultimate integral in the form (4.6) and that of the last integral in the form

$$m_{ijk} n_k \frac{\partial \delta u_i}{\partial x_j} = 2\varepsilon_{kil} n_k m_{ijp} n_j n_p \delta\psi_l + m_{ijk} n_i n_j n_k \delta(\varepsilon_{ls} n_l n_s) -$$

$$- \frac{\partial}{\partial x_k}\left[n_j n_p (m_{ipk} + m_{spk} n_s n_i) - n_k n_p (m_{ijp} + m_{sjp} n_s n_i)\right] n_j \delta u_i + \tag{5.2}$$

$$+ \varepsilon_{mrl} n_m \frac{\partial}{\partial x_r}\left[\varepsilon_{jlp} n_p n_k (m_{ijk} + m_{sjk} n_s n_i)\delta u_i\right]$$

Substituting (5.1), (5.2) and (4.6) into (2.1) and taking into account independence of variations δu_i in the volume v, of variations δu_i, $\delta\psi_l$, $\delta(\varepsilon_{ls} n_l n_s)$ on the surface s and of variations $\delta(t_l u_l)$, $\delta(u_i - u_l t_l t_i)$ at the kinks, we find the equations of equilibrium

$$\frac{\partial}{\partial x_j}\left(\sigma_{ij} + \frac{1}{2}\varepsilon_{ijl}\frac{\partial \mu_{kl}}{\partial x_k} - \frac{\partial m_{ijk}}{\partial x_k}\right) = 0 \tag{5.3}$$

the boundary conditions

$$\left\{\sigma_{ij} + \frac{1}{2}\varepsilon_{ijl}\left(\frac{\partial \mu_{kl}}{\partial x_k} - \frac{\partial \mu_{(n)}}{\partial x_l}\right) - \frac{\partial}{\partial x_k}\left[m_{ijk} + n_j n_p (m_{ipk} + m_{spk} n_s n_i) -\right.\right.$$

$$\left.\left. - n_k n_p (m_{ijp} + m_{sjp} n_s n_i)\right]\right\} n_j = q_i \tag{5.4}$$

$$(\mu_{kl} n_k - \mu_{(n)} n_l + 2\varepsilon_{kil} n_k m_{ijp} n_j n_p) = 0, \quad m_{ijk} n_i n_j n_k = 0$$

and the conditions at the kinks

$$\left[\left[\frac{1}{2}\mu_{(n)} + t_r \nu_j n_k (m_{rjk} + m_{sjk} n_s n_r)\right]\right] = 0$$

$$[[\nu_j n_k \{(m_{ijk} + m_{sjk} n_s n_i) - t_r t_i (m_{rjk} + m_{sjk} n_s n_r)\}]] = 0 \tag{5.5}$$

The same notation is used in (5.5) as in (2.9).

The relations (5.3) and (5.5) contain the equations of the couple stress theory of elasticity (4.7) to (4.9) as a particular case and for $m_{ijk} = 0$ become identical with them.

In this manner, displacements u_i which represent the solution of the couple stress theory of elasticity (4.7) to (4.9) determine mean displacements of a micro-inhomogeneous body, if only a part of the energy (1.5) corresponding to the strain tensor and to the anti-symmetrical part $\varkappa_{[ij]k}$ of the gradient of deformation tensor (3.1) is taken into account. Displacements u_i, presenting the solution of the boundary value problem (2.7) to (2.9) or (5.3) to (5.5) determine mean displacements of a micro-inhomogeneous body (1.1), taking into account the total energy (1.5). Therefore, from the point of view considered, the couple stress theory of elasticity (4.7) to (4.9) is only of a limited interest and the generalized couple stress theory of elasticity (2.7) to (2.9) or (5.3) to (5.5), appears to be more advantageous.

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